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The shear-layer structure in a rotating fluid near a differentially rotating sidewall

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This paper describes the flow of a homogeneous fluid contained in a rapidly rotating cylinder. The upper part of the cylinder rotates slightly faster, giving rise to a discontinuity in the sidewall velocity. The Stewartson-layer structure arising at the sidewall is essentially affected by this discontinuity. In contrast with previously studied problems, the $E^{\frac{1}{4}}$ layer (E is the Ekman number) is unable to perform the matching of the interior flow to the sidewall. It is shown that this matching is carried out partially by the $E^{\frac{1}{4}}$ layer and partially by the $E^{\frac{1}{3}}$ layer, the latter accounting for the jump discontinuity. This paper also presents an analytical description of the flow in the singularity region near the sidewall discontinuity.

1. Introduction

In recent years the flow of an incompressible fluid in a rapidly rotating container has been studied in a variety of configurations. An important feature in many flow configurations is the shear-layer structure arising at lateral flow boundaries. Shear layers of this type were first discussed by Stewartson (1957), who showed that they have a sandwich structure consisting of layers of thicknesses $E^{\frac{1}{4}}$ and $E^{\frac{1}{3}}$, where E is the Ekman number of the flow. Stewartson layers may be free, as in Stewartson's original problem and in the configurations studied by Moore & Saffman (1969), or attached to a solid wall (see e.g. Johnson 1974).

In all studies (involving attached as well as detached Stewartson layers) published thus far the matching of $O(1)$ velocities was entirely accomplished by the $E^{\frac{1}{4}}$ layer, while the $E^{\frac{1}{3}}$ layer only played a role in performing higher-order matching and providing vertical transport $O(E^{\frac{1}{2}})$. However, in the present paper we will investigate a flow configuration in which the $E^{\frac{1}{4}}$ layer is no longer able to accomplish the $O(1)$ matching completely.

The configuration is sketched in figure 1. Fluid is confined in a right circular cylinder of height HL and radius aL, which rotates about its (vertical) axis with constant angular velocity $\Omega_B = \Omega(1-\epsilon)$. The top disk and the upper section of the sidewall rotate slightly faster with angular speed $\Omega_T = \Omega(1+\epsilon)$, $0 < \epsilon \ll 1$, resulting in a jump discontinuity in the circumferential sidewall velocity. Throughout this study the fluid motion is related to a cylindrical coordinate system (r, θ, z) rotating at angular velocity Ω, with z pointing in the axial direction; the corresponding velocity components are denoted by (u, v, w). For mathematical convenience lengths and velocities are non-dimensionalized by L and $\epsilon \Omega L$ respectively.

The flow at some distance from the solid boundaries is in geostrophic balance and

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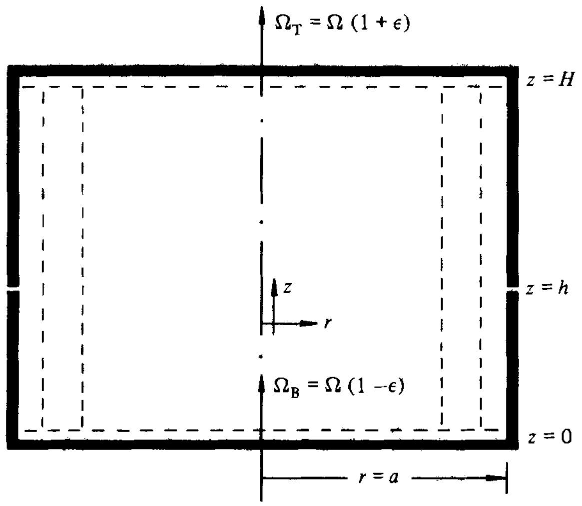

FIGURE 1. The geometry of the problem.

is entirely governed by the Ekman layers at the horizontal boundaries; a brief description of these flow regions will be given in §2. At the sidewall a Stewartson layer arises in order to bring the azimuthal interior velocity to relative rest. It is clear that the quasi-geostrophic $E^{\frac{1}{4}}$ layer is unable to account for this matching, which is in remarkable contrast with cases studied so far. As will be pointed out in §3, the non-geostrophic $E^{\frac{1}{3}}$ layer has to account for the jump in the wall velocity instead. It turns out, however, that the $E^{\frac{1}{3}}$ layer cannot accomplish a complete $O(1)$ matching either, but the combination of both layers leads to a correct matching.

Finally, the flow in the singularity associated with the gap in the sidewall is analysed in §4.

It should be mentioned that the flow in this configuration has also been considered by Hocking (1962) for the special case of an infinitely long cylinder (i.e. ignoring end effects). However, Hocking did not analyse the Stewartson-layer structure, which is the main objective of the present paper.

2. Formulation

The steady motion of an incompressible fluid relative to a reference system rotating with angular velocity Ω is described by

$$2\boldsymbol{k}\times\boldsymbol{v} = -\nabla p + E\nabla^2\boldsymbol{v}, \qquad (2.1a)$$

$$\nabla\cdot\boldsymbol{v} = 0. \qquad (2.1b)$$

Here $\boldsymbol{v}$, p and $\boldsymbol{k}$ are the velocity vector, the reduced pressure and a unit vector in the axial direction ($\boldsymbol{k} = \boldsymbol{\Omega}/\Omega$) respectively, while E is the Ekman number, defined as $E = \nu/\Omega L^2$ (ν being the kinematic viscosity). It is assumed that the relative flow is small enough that inertial effects may be neglected (i.e. the Rossby number $\epsilon \ll E^{\frac{1}{2}}$). Besides, we restrict ourselves to high rotation rates, i.e. $E \ll 1$.

In this paper we shall seek a solution $\boldsymbol{v}$ of (2.1) that satisfies the following boundary conditions:

$$\left.\begin{array}{ll} \boldsymbol{v} = (0, -r, 0) & (z = 0, \quad 0 < r < a), \\ \boldsymbol{v} = (0, -a, 0) & (0 < z < h, \quad r = a), \\ \boldsymbol{v} = (0, a, 0) & (h < z < H, \quad r = a), \\ \boldsymbol{v} = (0, r, 0) & (z = H, \quad 0 < r < a). \end{array}\right\} \tag{2.2}$$

At some distance from the solid boundaries the flow is in geostrophic balance, and is completely governed by the Ekman layers at the horizontal boundaries $z = 0$ and $z = H$. These Ekman layers provide the usual compatibility conditions, which enable one to derive the geostrophic velocity components:

$$u_{\mathrm{I}} = 0, \quad v_{\mathrm{I}} = 0, \quad w_{\mathrm{I}} = E^{\frac{1}{2}} \tag{2.3}$$

(the subscript I refers to the interior). According to these expressions, there is no interior motion relative to the rotating frame, except for a weak axial flow $O(E^{\frac{1}{2}})$ directed towards the faster-rotating disk.

The Ekman layers carry radial $O(E^{\frac{1}{2}})$ transport Q_{rad}, which can easily be calculated by integrating the radial velocity components over the layer thickness, yielding

$$Q_{\mathrm{rad}}(z = 0) = -\tfrac{1}{2} r E^{\frac{1}{2}}, \quad Q_{\mathrm{rad}}(z = H) = +\tfrac{1}{2} r E^{\frac{1}{2}}. \tag{2.4}$$

The Stewartson layer at the sidewall ($r = a$) has a dual task in satisfying the no-slip condition at the wall and producing a downward $O(E^{\frac{1}{2}})$ transport from upper to lower Ekman layer.

3. The shear-layer structure

As usual, the Stewartson layer at the sidewall ($r = a$) consists of sublayers of thicknesses $E^{\frac{1}{4}}$ and $E^{\frac{1}{3}}$. Because of its quasi-geostrophic character (i.e. the azimuthal velocity is independent of z), the $E^{\frac{1}{4}}$ layer is unable to perform the matching of the azimuthal interior velocity $v_{\mathrm{I}}(r = a) = 0$ to the discontinuous wall velocity V:

$$\left.\begin{array}{ll} V = -a & (0 < z < h), \\ \quad = +a & (h < z < H). \end{array}\right\} \tag{3.1}$$

This $O(1)$ matching must then necessarily be carried out by the $E^{\frac{1}{3}}$ layer. In addition, the Stewartson layer must carry a vertical transport $O(E^{\frac{1}{2}})$ from upper to lower Ekman layer; as will be shown hereinafter, both the $E^{\frac{1}{4}}$ layer and the $E^{\frac{1}{3}}$ layer participate in this.

3.1. *The $E^{\frac{1}{3}}$ layer: $n = 0$ field*

The velocity components and the pressure in the $E^{\frac{1}{3}}$ layer are expanded as

$$(u, v, w, p) = \sum_{n=0}^{\infty} E^{\frac{1}{12}n}(E^{\frac{1}{3}}\tilde{u}^{(n)}, \tilde{v}^{(n)}, \tilde{w}^{(n)}, E^{\frac{1}{3}}\tilde{p}^{(n)}),$$

and substitution into (2.1) yields for $n = 0, 1, 2, 3,$

$$\left.\begin{array}{r} -2\tilde{v}^{(n)} = -\tilde{p}_{\eta}^{(n)}, \\ 2\tilde{u}^{(n)} = \tilde{v}_{\eta\eta}^{(n)}, \\ 0 = -\tilde{p}_{z}^{(n)} + \tilde{w}_{\eta\eta}^{(n)}, \\ \tilde{u}_{\eta}^{(n)} + \tilde{w}_{z}^{(n)} = 0, \end{array}\right\} \tag{3.2}$$

with $\eta = (r-a)E^{-\frac{1}{3}}$. Elimination of $\tilde{p}^{(n)}$ and $\tilde{u}^{(n)}$ results in

$$2\tilde{v}_z^{(n)} = \tilde{w}_{\eta\eta\eta}^{(n)}, \quad 2\tilde{w}_z^{(n)} = -\tilde{v}_{\eta\eta\eta}^{(n)}. \tag{3.3}$$

Matching the azimuthal interior velocity to the sidewall requires $\tilde{v} = O(1)$, and hence $n = 0$. Then the boundary conditions for the $n = 0$ problem can be formulated as

$$\tilde{u}^{(0)}, \tilde{v}^{(0)}, \tilde{w}^{(0)} \to 0 \quad (\eta \to \infty), \tag{3.4a}$$

$$\left.\begin{aligned} &\tilde{u}^{(0)} = \tilde{w}^{(0)} = 0 \\ &\tilde{v}^{(0)} = -a \quad (0 < z < h) \\ &\tilde{v}^{(0)} = +a \quad (h < z < H) \end{aligned}\right\} \quad (\eta = 0), \tag{3.4b}$$

$$\tilde{w}^{(0)} = 0 \quad (z = 0, H). \tag{3.4c}$$

The $n = 0$ field essentially produces vertical transport $O(E^{\frac{1}{3}})$ which cannot be balanced by any other layer contribution. This would necessitate an additional condition for the $\tilde{w}^{(0)}$ solution requiring the total vertical $O(E^{\frac{1}{3}})$ transport to be zero. However, it can be shown by integrating the continuity equation (3.2*d*) with respect to η that this zero-transport requirement is already implied by the boundary conditions (3.4).

The general solutions of (3.2), (3.3), satisfying the Ekman suction conditions (3.4*c*), are

$$\left.\begin{aligned} \tilde{w}^{(0)} &= \sum_{n=1}^{\infty} (a_n \mathrm{e}^{\gamma_n \eta} + b_n \mathrm{e}^{-\omega\gamma_n\eta} + c_n \mathrm{e}^{-\omega^2\gamma_n\eta}) \sin\frac{n\pi z}{H}, \\ \tilde{v}^{(0)} &= -\sum_{n=1}^{\infty} (a_n \mathrm{e}^{\gamma_n \eta} - b_n \mathrm{e}^{-\omega\gamma_n\eta} - c_n \mathrm{e}^{-\omega^2\gamma_n\eta}) \cos\frac{n\pi z}{H}, \\ \tilde{u}^{(0)} &= -\sum_{n=1}^{\infty} \tfrac{1}{2}\gamma_n^2 (a_n \mathrm{e}^{\gamma_n \eta} - \omega^2 b_n \mathrm{e}^{-\omega\gamma_n\eta} - \omega c_n \mathrm{e}^{-\omega^2\gamma_n\eta}) \cos\frac{n\pi z}{H}, \end{aligned}\right\} \tag{3.5}$$

with $\gamma_n = (2n\pi/H)^{\frac{1}{3}}$, $\omega = -\frac{1}{2}+\frac{1}{2}\mathrm{i}\sqrt{3}$. By applying the no-slip conditions (3.4*b*), the coefficients a_n, b_n, c_n can be determined: no-slip of $\tilde{u}^{(0)}$ and $\tilde{w}^{(0)}$ requires

$$a_n + b_n + c_n = 0, \quad a_n - \omega^2 b_n - \omega c_n = 0, \tag{3.6}$$

while $\tilde{v}^{(0)}(\eta = 0)$ should satisfy

$$\tilde{v}^{(0)}(\eta = 0) = -\sum_{n=1}^{\infty} (a_n - b_n - c_n) \cos\frac{n\pi z}{H} = f(z), \tag{3.7a}$$

with

$$\left.\begin{aligned} f(z) &= -a \quad (0 < z < h), \\ &= +a \quad (h < z < H). \end{aligned}\right\} \tag{3.7b}$$

In order to evaluate the summation it is necessary to find a Fourier cosine-series representation for the function $f(z)$. Since a cosine series corresponds to an even function, $f(z)$ must be extended on the interval $-H < z < 0$, so that $f(-z) = f(z)$. By elementary Fourier analysis one obtains

$$f(z) = a\left(1 - \frac{2h}{H}\right) - \frac{4a}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin\frac{n\pi h}{H} \cos\frac{n\pi z}{H}. \tag{3.8}$$

Apart from the cosine series, this Fourier expansion of $f(z)$ contains a constant $a(1-2h/H)$. Hence, in view of (3.7) and (3.8), we must conclude that $\tilde{v}^{(0)}$ cannot satisfy a no-slip condition of the form (3.7). Nevertheless, the obvious choice

$$a_n - b_n - c_n = \frac{4a}{n\pi} \sin\frac{n\pi h}{H} \tag{3.9}$$

leads to

$$\left.\begin{aligned}\tilde{v}^{(0)}(\eta=0) &= -a-a\left(1-\frac{2h}{H}\right) \quad (0<z<h),\\ &= +a-a\left(1-\frac{2h}{H}\right) \quad (h<z<H).\end{aligned}\right\} \tag{3.10}$$

Although it accounts for the jump $2a$ in the azimuthal velocity, the $n = 0$ field of the $E^{\frac{1}{3}}$ layer is unable to satisfy the no-slip condition (3.4*b*) completely (except in the special case $h = \frac{1}{2}H$). As mentioned before, a Stewartson $E^{\frac{1}{4}}$ layer is quasi-geostrophic, and therefore not able to account for the jump in the swirl velocity. However, it can provide a matching to a constant 'wall velocity' V_0, so that (3.10) can be reformulated as

$$\left.\begin{aligned}\tilde{v}^{(0)}(\eta=0)+V_0 &= -a \quad (0<z<h),\\ &= +a \quad (h<z<H),\end{aligned}\right\} \tag{3.11a}$$

with

$$V_0 = a\left(1-\frac{2h}{H}\right). \tag{3.11b}$$

The coefficients a_n, b_n, c_n are then found from (3.6) and (3.9), yielding

$$\left.\begin{aligned}\tilde{w}^{(0)}(\eta,z) &= \frac{2a}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi h}{H}\sin\frac{n\pi z}{H}[e^{\gamma_n\eta}-\phi_1(\gamma_n;\eta)+\tfrac{1}{3}\sqrt{3}\,\phi_2(\gamma_n;\eta)],\\ \tilde{v}^{(0)}(\eta,z) &= -\frac{2a}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi h}{H}\cos\frac{n\pi z}{H}[e^{\gamma_n\eta}+\phi_1(\gamma_n;\eta)-\tfrac{1}{3}\sqrt{3}\,\phi_2(\gamma_n;\eta)],\\ \tilde{u}^{(0)}(\eta,z) &= -\frac{2a}{H}\sum_{n=1}^{\infty}\frac{1}{\gamma_n}\sin\frac{n\pi h}{H}\cos\frac{n\pi z}{H}[e^{\gamma_n\eta}-\phi_1(\gamma_n;\eta)-\tfrac{1}{3}\sqrt{3}\,\phi_2(\gamma_n;\eta)],\end{aligned}\right\} \tag{3.12}$$

with

$$\left.\begin{aligned}\phi_1(\alpha;\eta) &= e^{\frac{1}{2}\alpha\eta}\cos\tfrac{1}{2}\alpha\eta\sqrt{3},\\ \phi_2(\alpha;\eta) &= e^{\frac{1}{2}\alpha\eta}\sin\tfrac{1}{2}\alpha\eta\sqrt{3}.\end{aligned}\right\} \tag{3.13}$$

It is worth pointing out that the constant 'wall velocity' V_0 can also be derived in an alternative way, viz by integrating (3.11*a*) from $z = 0$ to $z = H$. Since $\tilde{v}^{(0)}(\eta = 0)$ is represented by a cosine series of the form (3.7*a*), the integral over $\tilde{v}^{(0)}(\eta = 0)$ vanishes, and hence

$$HV_0 = -ah + a(H-h),$$

or

$$V_0 = a\left(1-\frac{2h}{H}\right).$$

This is a general derivation of V_0, in which no detailed information about the $\tilde{v}^{(0)}$ solution structure is required.

It is obvious that the $n = 1$ field is absent in this particular shear-layer problem. The $n = 2$ and $n = 3$ fields, however, play an important role in producing vertical $O(E^{\frac{1}{3}})$ transport and satisfying higher-order matching conditions, and will be discussed later.

3.2. *The $E^{\frac{1}{4}}$ layer*

The velocity components and the pressure in the $E^{\frac{1}{4}}$ layer are expanded as

$$(u, v, w, p) = \sum_{n=0}^{\infty} E^{\frac{1}{4}n}(E^{\frac{1}{2}}U,\ V,\ E^{\frac{1}{4}}W,\ E^{\frac{1}{4}}P),$$

and by substitution into (2.1) one may derive the following set of relevant equations:

$$\left.\begin{aligned} &U_z = V_z = W_{zz} = 0, \\ &V_{\xi\xi\xi} + 2W_z = 0, \\ &U = \tfrac{1}{2}V_{\xi\xi}, \end{aligned}\right\} \qquad (3.14)$$

in which $\xi = (r-a)\,E^{-\frac{1}{4}}$ is the stretched radial coordinate. The Ekman suction conditions

$$W(z=0) = +\tfrac{1}{2}V_{\xi}, \quad w(z=H) = -\tfrac{1}{2}V_{\xi}$$

enable one to derive from (3.14)

$$V_{\xi\xi\xi} - \frac{2}{H}V_{\xi} = 0, \quad W = \left(\frac{1}{2} - \frac{z}{H}\right)V_{\xi}.$$

The V and W solutions can be determined easily, and, by requiring $V(\xi = 0) = a(1-2h/H)$ and $V(\xi \to -\infty) = 0$, one obtains

$$V(\xi) = a\left(1 - \frac{2h}{H}\right)\exp\left[\xi\left(\frac{2}{H}\right)^{\frac{1}{2}}\right], \qquad (3.15a)$$

$$W(\xi, z) = a\left(1 - \frac{2h}{H}\right)\left(\frac{2}{H}\right)^{\frac{1}{2}}\left(\frac{1}{2} - \frac{z}{H}\right)\exp\left[\xi\left(\frac{2}{H}\right)^{\frac{1}{2}}\right], \qquad (3.15b)$$

while the solution for the radial velocity is found from (3.14c):

$$U(\xi) = \frac{a}{H}\left(1 - \frac{2h}{H}\right)\exp\left[\xi\left(\frac{2}{H}\right)^{\frac{1}{2}}\right]. \qquad (3.15c)$$

These expressions show clearly that the $E^{\frac{1}{4}}$ layer is completely *absent* if $h = \frac{1}{2}H$; in this special case the $E^{\frac{1}{3}}$ layer is able to perform the $O(1)$ matching to the sidewall. On the other hand, if the gap lies in the bottom ($h = 0$) or top corner ($h = H$), the matching is entirely accomplished by the $E^{\frac{1}{4}}$ layer, the $E^{\frac{1}{3}}$ layer ($n = 0$ field) then being absent.

The local (i.e. per unit length of circumference) vertical $O(E^{\frac{1}{2}})$ transport $T(z)$ in the $E^{\frac{1}{4}}$ layer can be calculated by integrating W over the layer thickness:

$$T(z) = \int_{-\infty}^{0} W\,\mathrm{d}\xi = a\left(1 - \frac{2h}{H}\right)\left(\frac{1}{2} - \frac{z}{H}\right). \qquad (3.16)$$

By considering the total mass flow $O(E^{\frac{1}{2}})$ across a horizontal plane at arbitrary height z (including contributions from the interior, the $E^{\frac{1}{4}}$ layer and the $E^{\frac{1}{3}}$ layer), one derives the following condition for the vertical $O(E^{\frac{1}{2}})$ transport $\tilde{T}(z)$ in the $E^{\frac{1}{3}}$ layer:

$$\tilde{T}(z) = -a\left(1 - \frac{2h}{H}\right)\left(\frac{1}{2} - \frac{z}{H}\right) - \tfrac{1}{2}a. \qquad (3.17)$$

In the $E^{\frac{1}{3}}$ layer vertical transport $O(E^{\frac{1}{2}})$ is carried by the $n = 2$ field, which will be analysed next.

3.3. *The $E^{\frac{1}{3}}$ layer: $n = 2$ field*

In providing a vertical $O(E^{\frac{1}{2}})$ transport as given by (3.17) and matching $U(\xi)$ to the sidewall, the $n = 2$ field of the $E^{\frac{1}{3}}$ layer plays a dual role. The boundary conditions for this problem are

$$\left.\begin{aligned} &\tilde{u}^{(2)}, \tilde{v}^{(2)}, \tilde{w}^{(2)} \to 0 \quad (\eta \to -\infty), \\ &\left.\begin{aligned} &\tilde{v}^{(2)} = \tilde{w}^{(2)} = 0, \\ &\tilde{u}^{(2)} = -U(\xi = 0) = -\frac{a}{H}\left(1 - \frac{2h}{H}\right) \end{aligned}\right\} (\eta = 0); \end{aligned}\right\} \tag{3.18a}$$

$$\left.\begin{aligned} \tilde{w}^{(2)} &= +\frac{1}{2}\frac{\partial \tilde{v}^{(0)}}{\partial \eta}\bigg|_{z=0} + C_{\mathrm{B}}\delta(\eta) \quad (z = 0), \\ \tilde{w}^{(2)} &= -\frac{1}{2}\frac{\partial \tilde{v}^{(0)}}{\partial \eta}\bigg|_{z=H} + C_{\mathrm{T}}\delta(\eta) \quad (z = H); \end{aligned}\right\} \tag{3.18b}$$

$$\int_{-\infty}^{0} \tilde{w}^{(2)}\, \mathrm{d}\eta = -\tfrac{1}{2}a - a\left(1 - \frac{2h}{H}\right)\left(\frac{1}{2} - \frac{z}{H}\right) \quad (0 < z < H). \tag{3.18c}$$

In the Ekman suction conditions (3.18*b*) δ-functions are included in order to be able to describe possible singular flow associated with the $O(E^{\frac{1}{2}} \times E^{\frac{1}{3}})$ corner regions at $z = 0$ and $z = H$. By substitution of the $\tilde{v}^{(0)}$ solution (3.12*b*), the suction conditions become

$$\left.\begin{aligned} \tilde{w}^{(2)}(z = 0) &= -\frac{a}{\pi}\sum_{n=1}^{\infty}\frac{\gamma_n}{n}\sin\frac{n\pi h}{H}[\mathrm{e}^{\gamma_n\eta} - \tfrac{2}{3}\sqrt{3}\,\phi_2(\gamma_n;\eta)] + C_{\mathrm{B}}\delta(\eta), \\ \tilde{w}^{(2)}(z = H) &= +\frac{a}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n\gamma_n}{n}\sin\frac{n\pi h}{H}[\mathrm{e}^{\gamma_n\eta} - \tfrac{2}{3}\sqrt{3}\,\phi_2(\gamma_n;\eta)] + C_{\mathrm{T}}\delta(\eta), \end{aligned}\right\} \tag{3.19}$$

and integration with respect to η yields

$$\left.\begin{aligned} \int_{-\infty}^{0} \tilde{w}^{(2)}(z = 0)\, \mathrm{d}\eta &= -\frac{2a}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi h}{H} + C_{\mathrm{B}} = -a\left(1 - \frac{h}{H}\right) + C_{\mathrm{B}}, \\ \int_{-\infty}^{0} \tilde{w}^{(2)}(z = H)\, \mathrm{d}\eta &= +\frac{2a}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi h}{H} + C_{\mathrm{T}} = -\frac{ah}{H} + C_{\mathrm{T}}. \end{aligned}\right\} \tag{3.20}$$

The general transport condition (3.18*c*) requires

$$C_{\mathrm{B}} = C_{\mathrm{T}} = 0,$$

which implies that there are no corner singularities in the general case $0 < h < H$. In the special cases $h = 0$ and $h = H$, however, the situation is remarkably different: the $n = 0$ field of the $E^{\frac{1}{3}}$ layer then being absent, the suction conditions (3.18*b*) are

$$\tilde{w}^{(2)}(z = 0) = C_{\mathrm{B}}\delta(\eta), \quad \tilde{w}^{(2)}(z = H) = C_{\mathrm{T}}\delta(\eta), \tag{3.21}$$

with C_{B} and C_{T} being determined from (3.18*c*):

$$\left.\begin{aligned} C_{\mathrm{B}} &= -2a, \quad C_{\mathrm{T}} = 0 \quad (h = 0), \\ C_{\mathrm{B}} &= 0, \quad C_{\mathrm{T}} = -2a \quad (h = H). \end{aligned}\right\} \tag{3.22}$$

Solutions of (3.3) satisfying the conditions (3.18*a*), (3.21) and (3.22) can be obtained by standard techniques, yielding

$$\left.\begin{aligned} \tilde{w}^{(2)}(\eta,z) &= \sum_{n=1}^{\infty} A_n\,\phi_2(\gamma_n;\eta)\sin\frac{n\pi z}{H},\\ \tilde{v}^{(2)}(\eta,z) &= -\sum_{n=1}^{\infty} A_n\,\phi_2(\gamma_n;\eta)\cos\frac{n\pi z}{H},\\ \tilde{u}^{(2)}(\eta,z) &= \frac{1}{4}\sum_{n=1}^{\infty}\gamma_n^2 A_n[\surd 3\,\phi_1(\gamma_n;\eta)+\phi_2(\gamma_n;\eta)]\cos\frac{n\pi z}{H}, \end{aligned}\right\} \tag{3.23}$$

with

$$A_n = \frac{4[C_{\mathrm{B}}-(-1)^n C_{\mathrm{T}}]}{H\gamma_n^2\surd 3}, \quad \gamma_n = \left(\frac{2n\pi}{H}\right)^{\frac{1}{3}}.$$

However, in the general case $0 < h < H$, one has to seek solutions that satisfy the suction conditions (3.19) with $C_{\mathrm{B}} = C_{\mathrm{T}} = 0$. A similar problem has been studied by Hunter (1967), and his approach will be followed here.

By introduction of the stream function $\tilde{\psi}(\eta,z)$,

$$\tilde{u}^{(2)} = \tilde{\psi}_z, \quad \tilde{w}^{(2)} = -\tilde{\psi}_\eta \rightarrow \tilde{\psi}(\eta,z) = -\int_{-\infty}^{\eta}\tilde{w}^{(2)}(x,z)\,\mathrm{d}x, \tag{3.24}$$

(3.3) can be rearranged to

$$\frac{\partial^6\tilde{\psi}}{\partial\eta^6}+4\frac{\partial^2\tilde{\psi}}{\partial z^2} = 0. \tag{3.25}$$

In view of the transport condition (3.18*c*) we take $\tilde{\psi} = \tilde{\psi}_1+\tilde{\psi}_2$, with

$$\left.\begin{aligned} \tilde{\psi}_1(0,z) &= \tfrac{1}{2}a,\\ \tilde{\psi}_2(0,z) &= a\left(1-\frac{2h}{H}\right)\left(\frac{1}{2}-\frac{z}{H}\right). \end{aligned}\right\} \tag{3.26}$$

The conditions (3.19) at $z = 0$ and $z = H$ require

$$\left.\begin{aligned} \tilde{\psi}_1(\eta,0) &= +\tilde{\psi}_1(\eta,H) = \tilde{A}(\eta),\\ \tilde{\psi}_2(\eta,0) &= -\tilde{\psi}_2(\eta,H) = \tilde{B}(\eta), \end{aligned}\right\} \tag{3.27}$$

with

$$\tilde{A}(\eta) = \frac{a}{2\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin\frac{n\pi h}{H}F(\eta),$$

$$\tilde{B}(\eta) = \frac{a}{2\pi}\sum_{n=1}^{\infty}\frac{1+(-1)^n}{n}\sin\frac{n\pi h}{H}F(\eta),$$

$$F(\eta) = \mathrm{e}^{\gamma_n\eta}+\phi_1(\gamma_n;\eta)-\tfrac{1}{3}\surd 3\,\phi_2(\gamma_n;\eta).$$

It can easily be verified that $\tilde{A}(0) = \frac{1}{2}a$ and $\tilde{B}(0) = \frac{1}{2}a(1-2h/H)$. In summary, the boundary conditions for the $\tilde{\psi}_1, \tilde{\psi}_2$ problems are

$$\left.\begin{aligned} \tilde{\psi}_1 &= \tilde{A}(\eta) \quad (z=0),\\ \tilde{\psi}_1 &= \tilde{A}(\eta) \quad (z=H); \end{aligned}\right\} \tag{3.28a}$$

$$\tilde{\psi}_1 = \tfrac{1}{2}a, \quad \frac{\partial\tilde{\psi}_1}{\partial\eta} = 0, \quad \tilde{v}_1^{(2)} = 0 \quad (\eta = 0); \tag{3.28b}$$

$$\left.\begin{aligned} \tilde{\psi}_2 &= +\tilde{B}(\eta) \quad (z=0),\\ \tilde{\psi}_2 &= -\tilde{B}(\eta) \quad (z=H); \end{aligned}\right\} \tag{3.29a}$$

$$\tilde{\psi}_2 = a\left(1-\frac{2h}{H}\right)\left(\frac{1}{2}-\frac{z}{H}\right), \quad \frac{\partial\tilde{\psi}_2}{\partial\eta} = 0, \quad \tilde{v}_2^{(2)} = 0 \quad (\eta = 0). \tag{3.29b}$$

The general $\tilde{\psi}_1$ solution satisfying the conditions at $z = 0$ and $z = H$ is

$$\tilde{\psi}_1(\eta, z) = \tilde{A}(\eta) + \sum_{n=0}^{\infty} f_n(\eta) \sin\frac{n\pi z}{H}, \tag{3.30}$$

and substitution into (3.25) yields an equation for $f_n(\eta)$:

$$\frac{d^6 f_n}{d\eta^6} - \gamma_n^6 f_n(\eta) = -\frac{4}{H\gamma_n^3}[1-(-1)^n]\tilde{A}^{\mathrm{vi}}(\eta), \quad \gamma_n^3 = \frac{2n\pi}{H}. \tag{3.31}$$

By integrating $\tilde{v}_z = -\frac{1}{2}\tilde{\psi}_{\eta\eta\eta\eta}$ with respect to z, the $\tilde{v}^{(2)}$ solution is found:

$$\tilde{v}_1^{(2)} = -\tfrac{1}{2}z\,\tilde{A}^{\mathrm{iv}}(\eta) + \sum_{n=0}^{\infty} \frac{1}{\gamma_n^3} f_n^{\mathrm{iv}}(\eta) \cos\frac{n\pi z}{H}.$$

The conditions (3.28*b*) require

$$f_n(0) = 0, \quad f_n'(0) = -\frac{4}{H\gamma_n^3}[1-(-1)^n]A'(0), \quad f_n^{\mathrm{iv}}(0) = -\frac{4}{H\gamma_n^3}[1-(-1)^n]\tilde{A}^{\mathrm{iv}}(0),$$

and the solution of (3.31) that satisfies these boundary conditions and tends to zero as $\eta \to -\infty$ can be obtained as pointed out by Hunter (1967):

$$\left.\begin{aligned} f_{2n}(\eta) &= 0,\\ f_{2n+1}(\eta) &= -\frac{4\tilde{A}'(0)}{H\alpha_n^4}[e^{\alpha_n\eta} - \phi_1(\alpha_n;\eta) + \sqrt{3}\,\phi_2(\alpha_n;\eta)]\\ &\quad -\frac{4\tilde{A}^{\mathrm{iv}}(0)}{H\alpha_n^7}[e^{\alpha_n\eta} - \phi_1(\alpha_n;\eta) - \tfrac{1}{3}\sqrt{3}\,\phi_2(\alpha_n;\eta)] + \mathscr{L}\left(\alpha_n, -\frac{8\tilde{A}^{\mathrm{vi}}(\eta)}{H\alpha_n^3}\right),\\ \alpha_n^3 &= \gamma_{2n+1}^3 = \frac{2(2n+1)\pi}{H}. \end{aligned}\right\} \tag{3.32}$$

Here $\mathscr{L}(\alpha, \varphi)$ is the solution of the sixth-order non-homogeneous equation

$$\frac{d^6 y}{d\eta^6} - \alpha^6 y = \varphi(\eta)$$

that vanishes as $\eta \to -\infty$ and satisfies $y = y_\eta = y_{\eta\eta\eta\eta} = 0$ at $\eta = 0$; this solution $\mathscr{L}(\alpha, \varphi)$ is similar to that given by Hunter (1967, equation (36)).

By analogy, the general $\tilde{\psi}_2$ solution is

$$\tilde{\psi}_2(\eta, z) = 2\left(\frac{1}{2}-\frac{z}{H}\right)\tilde{B}(\eta) + \sum_{n=0}^{\infty} g_n(\eta) \sin\frac{n\pi z}{H} \tag{3.33}$$

and $g_n(\eta)$ can be determined as above, yielding

$$\left.\begin{aligned} g_{2n+1}(\eta) &= 0,\\ g_{2n}(\eta) &= -\frac{4\tilde{B}'(0)}{H\beta_n^4}[e^{\beta_n\eta} - \phi_1(\beta_n;\eta) + \sqrt{3}\,\phi_2(\beta_n;\eta)]\\ &\quad -\frac{4\tilde{B}^{\mathrm{iv}}(0)}{H\beta_n^7}[e^{\beta_n\eta} - \phi_1(\beta_n;\eta) - \tfrac{1}{3}\sqrt{3}\,\phi_2(\beta_n;\eta)] + \mathscr{L}\left(\beta_n, \frac{-8\tilde{B}^{\mathrm{vi}}(\eta)}{H\beta_n^3}\right),\\ \beta_n^3 &= \gamma_{2n}^3 = \frac{4n\pi}{H}. \end{aligned}\right\} \tag{3.34}$$

This completes the analysis of the $n = 2$ field of the $E^{\frac{1}{3}}$ layer: the functions $f_n(\eta)$ and $g_n(\eta)$ and consequently the stream functions $\tilde{\psi}_1$ and $\tilde{\psi}_2$ are now determined. The total stream function is $\tilde{\psi} = \tilde{\psi}_1 + \tilde{\psi}_2$, from which the velocity components can be derived by use of the definition (3.24).

3.4. *The $E^{\frac{1}{3}}$ layer: $n = 3$ field*

The $n = 3$ field of $E^{\frac{1}{3}}$ layer plays a role in matching $W(\xi = 0)$ to the sidewall; the appropriate boundary conditions are

$$\left.\begin{aligned} &\tilde{u}^{(3)}, \tilde{v}^{(3)}, \tilde{w}^{(3)} \to 0 \quad (\eta \to -\infty); \\ &\left.\begin{aligned} &\tilde{u}^{(3)} = \tilde{v}^{(3)} = 0, \\ &\tilde{w}^{(3)} = -W(\xi = 0) = -a\left(1 - \frac{2h}{H}\right)\left(\frac{2}{H}\right)^{\frac{1}{2}}\left(\frac{1}{2} - \frac{z}{H}\right) \end{aligned}\right\} \quad (\eta = 0); \\ &\tilde{w}^{(3)} = 0 \quad (z = 0, H). \end{aligned}\right\} \tag{3.35}$$

Again, the general solutions are represented by (3.5), the coefficients to be determined by applying the conditions at $\eta = 0$; this results in

$$\left.\begin{aligned} \tilde{w}^{(3)}(\eta, z) &= \sum_{n=1}^{\infty} B_n[e^{\gamma_n \eta} + \phi_1(\gamma_n; \eta) + \sqrt{3}\, \phi_2(\gamma_n; \eta)] \sin \frac{n\pi z}{H}, \\ \tilde{v}^{(3)}(\eta, z) &= -\sum_{n=1}^{\infty} B_n[e^{\gamma_n \eta} - \phi_1(\gamma_n; \eta) - \sqrt{3}\, \phi_2(\gamma_n; \eta)] \cos \frac{n\pi z}{H}, \\ \tilde{u}^{(3)}(\eta, z) &= -\tfrac{1}{2} \sum_{n=1}^{\infty} B_n \gamma_n^2[e^{\gamma_n \eta} - \phi_1(\gamma_n; \eta) + \sqrt{3}\, \phi_2(\gamma_n; \eta)] \cos \frac{n\pi z}{H}, \end{aligned}\right\} \tag{3.36}$$

with

$$B_n = -\frac{a}{2n\pi}[1 + (-1)^n]\left(1 - \frac{2h}{H}\right)\left(\frac{2}{H}\right)^{\frac{1}{2}},$$

and with ϕ_1, ϕ_2 given by (3.13).

4. The singularity structure

The wall velocity changes discontinuously across the gap at $(r = a, z = h)$, and since the vertical scale of this gap is assumed to be small, i.e. $\ll O(1)$, the flow near it will not be governed by the $E^{\frac{1}{3}}$ layer equations (3.2). Instead, one should apply the more complete equations

$$-2v = -\frac{\partial p}{\partial r} + E\nabla^2 u, \quad 2u = E\nabla^2 v. \tag{4.1 a, b}$$

By assuming

$$\frac{\partial}{\partial r}, \frac{\partial}{\partial z} \sim O(E^{\frac{1}{2}}), \quad \bar{u} \ll \bar{v} = O(1) \tag{4.2}$$

one finds $2\bar{u} \ll E\nabla^2 \bar{v} = O(1)$, or equivalently

$$\nabla^2 \bar{v} = \left(\frac{\partial^2}{\partial \rho^2} + \frac{\partial^2}{\partial \zeta^2}\right)\bar{v} = 0, \tag{4.3}$$

where $\rho = (r - a)E^{-\frac{1}{2}}$ and $\zeta = (z - h)E^{-\frac{1}{2}}$ are the stretched radial and vertical coordinates respectively (see figure 2). From (4.1 a) one derives $\bar{p} = O(E^{\frac{1}{2}})$, and hence

$$2\bar{v} = \frac{\partial \bar{p}}{\partial \rho},$$

showing that the radial Coriolis force is balanced by the radial pressure gradient only.

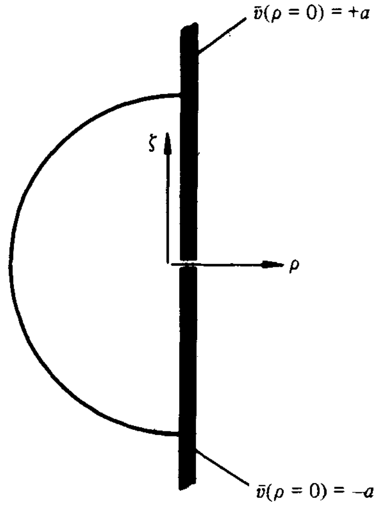

FIGURE 2. The geometry of the singularity at the gap.

The solution of (4.3) must satisfy the following boundary conditions:

$$\left.\begin{aligned} &\bar{v}\to 0 \quad (\rho\to-\infty),\\ &\bar{v}=-a \quad (\rho=0,\quad \zeta<0),\\ &\bar{v}=+a \quad (\rho=0,\quad \zeta>0).\end{aligned}\right\} \tag{4.4}$$

A similar problem with a jump in the boundary conditions has been studied by Steketee (1966), who showed that solutions of the form

$$\bar{v} \sim \tan^{-1}\frac{\rho}{\zeta}, \quad \tan^{-1}\frac{\zeta}{\rho}$$

are able to account for a jump discontinuity. In the present problem only the latter solution is relevant, and applying the conditions at $\rho = 0$ yields

$$\bar{v} = \frac{2a}{\pi}\tan^{-1}\frac{\zeta}{\rho}. \tag{4.5}$$

Because $\bar{v}(\rho\to-\infty) = 0$ and $\bar{v}(\zeta\to\pm\infty) = \pm a$, this solution presents a correct description of the flow in the vicinity of the gap in the sidewall.

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